OLD AND NEW RESULTS FOR SUPERENERGY TENSORS
FROM
DIMENSIONALLY DEPENDENT TENSOR IDENTITIES

S. Brian Edgar and Ola Wingbrant
Department of Mathematics,
Linköpings universitet,
Linköping,
Sweden S-581 83.

email: bredg@mai.liu.se, olawi214@student.liu.se

Abstract.
It is known that some results for spinors, and in particular for superenergy spinors, are much less transparent and require a lot more effort to establish, when considered from the tensor viewpoint. In this paper we demonstrate how the use of dimensionally dependent tensor identities enables us to derive a number of 4-dimensional identities by straightforward tensor methods in a signature independent manner. In particular, we consider the quadratic identity for the Bel-Robinson tensor $\mathcal{T}_{abcx}\mathcal{T}^{abcy} = \delta_x^y\ \mathcal{T}_{abcd}\mathcal{T}^{abcd}/4$ and also the new conservation laws for the Chevreton tensor, both of which have been obtained by spinor means; both of these results are rederived by *tensor* means for 4-dimensional spaces of any signature, using dimensionally dependent identities, and also we are able to conclude that there are no *direct* higher dimensional analogues. In addition we demonstrate a simple way to show non-existense of such identities via counter examples; in particular we show that there is no non-trivial Bel tensor analogue of this simple Bel-Robinson tensor quadratic identity. On the other hand, as a sample of the power of generalising dimensionally dependent tensor identities from four to higher dimensions, we show that the symmetry structure, trace-free and divergence-free nature of the four dimensional Bel-Robinson tensor does have an analogue for a class of tensors in higher dimensions.

1 Introduction.

Investigations connected with the Bel-Robinson tensor [1] in four dimensions are usually much simpler and more efficient when carried out in spinor formalism [2]. Senovilla [3] has demonstrated that a much larger class of tensors — *superenergy tensors* — share most of the desirable properties of the Bel-Robinson tensor, and Bergqvist [4] has shown how *superenergy spinors* give a simpler and more efficient presentation of certain aspects of these superenergy tensors. Recently, Bergqvist, Eriksson and Senovilla [5] have obtained new conservation laws for the electromagnetic field using superenergy spinor considerations, and emphasised that the proof of this result is far from obvious from the tensor point of view.

It is reassuring to know that certain important but perhaps unexpected properties — disguised in the complexities of the tensor formalism — become more transparent in the spinor formalism; but the parallel and more transparent spinor investigations are restricted to 4-dimensional spacetimes with Lorentz signature, and so this assistance is not available in higher dimensions nor in four dimensional spaces with other signatures. Deser [6] has emphasised the significance of the Bel-Robinson tensor in higher dimensions, and one of the important features of Senovilla's method of construction of superenergy tensors [3] is that it is applicable to arbitrary fields *in any dimension*; and so, for higher dimensions, it becomes an obvious concern whether there could be unexpected properties for superenergy tensors — disguised in the even deeper complexities of tensor formalism in higher dimensions — analogous to those properties revealed by spinor formalism in four dimensions.

Deeper investigations into the interaction between dimension and tensor identities have been instrumental in illustrating the uniqueness of some of the Bel-Robinson tensor's properties in four dimensions [6], explaining

the collapse of some Riemann scalar invariants in four dimensions [7], resolving apparent disparities between the spinor and tensors versions of the wave equations for the Weyl tensor and Lanczos potential [8,9,10,11] respectively. Moreover, in higher dimensions, worries concerning counterterms in Lagrangians [12,13] have been dispelled, and the Bel-Robinson tensor has been shown to be fully symmetric in five dimensions (as well as in four dimensions) [3].

Much earlier, Lovelock [14] had pointed out that a number of apparently unrelated results were all really consequences of a class of identities which he christened *dimensionally dependent identities* — identities which are a trivial, but subtle, consequence of dimension alone. Recently Edgar and Höglund [15] have generalised Lovelock's results, and demonstrated that the underlying principle in all of these investigations in [6 - 13], and some new ones, was the explicit exploitation of dimensionally dependent identities. Furthermore, in algebraic Rainich theory, Bergqvist and Höglund [16] have exploited these ideas further, and obtained results in *five* dimensions involving *cubic* terms in the energy momentum tensor — motivated by the familiar results in *four* dimensions involving *quadratic* terms, [17]; while Edgar and Höglund [18] have demonstrated the crucial role that dimensionally dependent identities play in the existence of the Lanczos potential for the Weyl tensor in different dimensions.

Deser [19] has applied the adjective 'ubiquitous' to the Bel-Robinson tensor, and this description is equally appropriate to dimensionally dependent identities as can be seen by the wide range of the investigations in [6 - 13], the applications given by Lovelock [14], and the more recent applications in [15,16,18]. In fact, Deser has argued elsewhere [6], that two identities (which are examples of what we call dimensionally dependent identities in four dimensions) are in a sense implicit in the familiar definitions of the Bel-Robinson tensor in four dimensions.

The purpose of this present paper is to emphasise the subtle interaction between dimension and tensor identities, and illustrate the important role which can be played by *fundamental dimensionally dependent identities* in investigations where tensor identities are important, in particular involving superenergy tensors. (We shall refer to the most fundamental dimensionally dependent identities as 'fddis', and to any identities constructed from these as 'ddis'). Our overall aim is to examine useful and significant properties in four dimensions — usually originating as spinor identities — and identify the kernel 4-dimensional fddi; then we will use the higher dimensional analogues of the kernel fddi to try and establish analogous results in higher dimensions. This will involve two different stages of investigation:

Step 1. The first step is to establish 4-*dimensional signature-independent tensor versions and proofs* of interesting spinor identities and/or reconcile apparent discrepencies between spinor and tensor results. We stress the need for *signature-independent* proofs for the following reason: results obtained using spinors strictly can claim to be valid only in 4-dimensional spacetimes *with Lorentz signature*. Of course there are results in such spaces which have no counterpart in other signatures (e.g., results concerning principle null directions of the Weyl tensor), but we encounter an uncertain situation when we consider results which can be stated in tensors with no apparent reference to signature, but which were derived in spinors, or derived in tensors but using features which are signature dependent[†]. The familiar identity for the Bel-Robinson tensor

$$\mathcal{T}_{abcx}\mathcal{T}^{abcy} = \delta_x^y \ \mathcal{T}_{abcd}\mathcal{T}^{abcd}/4 \tag{1}$$

in four dimensions is an important example of such a situation; we would wish to understand whether dimension and signature have crucial roles in this result[‡].

[†] When some results, which had been obtained by spinor means for the Weyl spinor [12] and Lanczos spinor [20], proved difficult to reproduce by tensor analysis there was speculation in [12] and [20] respectively that these results could be obtained *only* by spinor means. However, in these particular cases, signature independent versions were obtained by use of 4-dimensional tensor fddis. in [13,15] and [11] respectively.

[‡] Identity (1) is quoted in [21] with a reference to a proof by Debever in [22]; this proof is a lot more complicated than the spinor proof in [2]. Moreover, Debever's proof is also explicitly for a Lorentz spacetime; he makes use of the principle null directions, and it is not easy to see how this proof could be generalised to other signatures of 4-dimensional spaces, or to higher dimensions. Hence the identity has really only been proven in [2] and [22] for 4-dimensional spacetimes *with Lorentz signature*, and strictly its applicability to other signatures has not been confirmed in those proofs.

In spinor calculations the dimension four is inbuilt into the formalism; in tensor calculations a non-arbitrary dimension such as four has to be put in explicitly 'by hand'. However, it is not always sufficient just to substitute $n = 4$ in explicit calculations; in some cases the substitution needed is more subtle — it is achieved by the use of one or more fddis, but it is clear that there is no direct spinor analogue of a 4-dimensional fddi — the spinor version is trivially zero.

So, for example, the Lanczos spinor potential for the Weyl spinor $L_{ABCD'} = L_{(ABC)D'}$, in Ricci flat spaces, was found by Illge [10] to satisfy the very simple equation

$$\square L_{ABCD'} = 0 \tag{2}$$

while the corresponding tensor equation for the Lanczos tensor $L_{abc} = L_{[ab]c}$, $L^c{}_{ac} = 0 = L_{[abc]}$ is calculated to be [9,11]

$$\nabla^2 L_{abc} + \frac{2(n-4)}{n-2} L_{[a}{}^{d}{}_{|c;d|b]} = 2L_{[b}{}^{ed}C_{a]dec} - \frac{1}{2}C_{deab}L^{de}{}_c + \frac{4}{n-2}g_{c[a}C_{b]fed}L^{fed} \tag{3}$$

where C_{abcd} is the Weyl tensor. In four dimensions, obviously it cannot be sufficient simply to substitute $n = 4$, since we know from the spinor version that the right hand side must disappear completely in four dimensions. But if we consider the 4-dimensional fddi $C_{[ab}{}^{[cd}\delta^{e]}_{f]} \equiv 0$ [14,15] (quoted in Lemma 3 at the end of this section), we find that, when contracted with $L_{de}{}^f$, we obtain the ddi,

$$2L_{[b}{}^{de}C_{a]edc} - \frac{1}{2}L^{de}{}_c C_{deab} + 2L^{def}g_{c[a}C_{b]def} \equiv 0 \tag{4}$$

which ensures that the whole of the right hand side of (3) disappears in four dimensions [10].

Step 2. Once the 4-dimensional version is fully understood and the 4-dimensional kernel fddi obtained, the second step will be to determine what generalisations are possible using the higher dimensional counterpart fddis of the 4-dimensional kernel fddi. Occasionally these generalisations can be quite straightforward (e.g., discovering that a 4-dimensional result is also valid in five dimensions [3,15]); or more complicated involving a restructuring of the 4-dimensional result in higher dimensions (e.g., finding a 5-dimensional result involving triple products of Maxwell tensors as a generalisation of a 4-dimensional result involving double products of Maxwell tensors [16]).

The remainder of the paper is organised as follows. In Section 2 we deduce four different spinor identities which are special cases of one very simple general spinor identity; but in Section 3 we find that the 4-dimensional signature-independent tensor version of each of these identities requires a very different *tensor* proof — some of which are very complicated — although the unifying characteristic in all is the use of ddis. In Section 4 we show that the Bel superenergy tensor does not satisfy the same simple identity (1) in four dimensions, and in Section 5 we show that the Bel-Robinson tensor does not satisfy any analogous identity to (1) in *five* dimensions. In Section 7 we rederive the new conservation laws for electromagnetic theory [5] by using a number of 4-dimensional tensor ddis; these 4-dimensional tensor ddis cannot be replaced *directly* with higher dimensional ddis, and so there is no *direct* higher dimensional analogue of this law.

As noted above, the second step in such investigations is to attempt generalisation of 4-dimensional results to higher dinensions once we have identified the kernel tensor fddi in four dimensions. In Section 6 we give the 4-dimensional tensor counterparts to two trivial spinor results involving the symmetry properties of the Bel-Robinson and Lanczos superenergy spinors: these results both involve 4-dimensional fddis, and in the case of the Bel-Robinson superenergy tensor, by means of the analogous 5-dimensional fddi we show that exactly the same result is true in five dimensions.

A more ambitious generalisation is proposed in Section 8. We illustrate this approach by considering a superenergy tensor which is a natural generalisation of the Bel-Robinson tensor, and show that it shares its attractive properties of full index symmetry and zero divergence in *seven and lower* dimensions, as well as being trace-free in *six* dimensions. A summary is given, and future developments are proposed in Section 9.

It will be useful to have for reference a number of lemmas which are simply tensor ddis in four dimensions. Many familiar identities ostensibly involve the Weyl or Riemann tensors directly or indirectly, but on closer

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inspection have a more general character being simply algebraic, involving 'candidates'[†] for Weyl, Riemann, Lanczos or other tensors. In this paper, we shall give the results for the more general 'candidates' where appropriate.

The following lemmas can be found in [15], or can be deduced from results there. The first three of these lemmas are fddis obtained by skew symmetrising over *five* indices in 4-dimensional space (and *six* indices in 5-dimensional space), and exploiting the fact that the appropriate tensors are trace-free; Lemma 4 gives ddis deduced from Lemma 3, but as can be seen from the details in [23], although (8a) is well-known, quite a lot of work is involved in obtaining (8b,c,d).

Lemma 1. In four dimensions, a 2-tensor A_{ab} satisfies

$$A_{[a}{}^a A_b{}^b A_c{}^c A_d{}^d \delta^f_{e]} \equiv 0 \ , \tag{5}$$

which is equivalent to the Cayley-Hamilton Theorem for the 4×4 matrix $A_a{}^b$ when written out term by term.

(We shall be concerned with two special cases from this class of tensors: trace-free Ricci candidates $\hat{S}_{ab} = \hat{S}_{(ab)}$ with $\hat{S}^a{}_a = 0$, and Maxwell tensors $F_{ab} = F_{[ab]}$.)

Lemma 2. A Lanczos candidate $\hat{L}_{abc}$ with properties $\hat{L}_{abc} = \hat{L}_{[ab]c}$, $\hat{L}^c{}_{ac} = 0 = \hat{L}_{[abc]}$, satisfies

$$\hat{L}_{[ab}{}^{[e} \delta^{fg]}_{cd]} \equiv 0 \qquad \text{in four dimensions}, \tag{6a}$$

$$\hat{L}_{[ab}{}^{[f} \delta^{ghi]}_{cde]} \equiv 0 \qquad \text{in five dimensions}. \tag{6b}$$

Lemma 3. A Weyl candidate $\hat{C}_{ab}{}^{cd}$ satisfies

$$\hat{C}_{[ab}{}^{[de} \delta^{f]}_{c]} \equiv 0 \qquad \text{in four dimensions}, \tag{7a}$$

$$\hat{C}_{[ab}{}^{[ef} \delta^{gh]}_{cd]} \equiv 0 \qquad \text{in five dimensions}. \tag{7b}$$

Lemma 4. In four dimensions, a Weyl candidate $\hat{C}_{ab}{}^{cd}$ satisfies

$$\text{(a)} \qquad \hat{C}_{abcx}\hat{C}^{abcy} \equiv \delta^y_x \ \hat{C}_{abcd}\hat{C}^{abcd}/4 \ , \tag{8a}$$

$$\text{(b)} \qquad \hat{C}^{yb}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fc}{}_{ex} \equiv \delta^y_x \ \hat{C}^{ab}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fc}{}_{ea}/4 \ , \tag{8b}$$

$$\text{(c)} \qquad \hat{C}^{yb}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fg}{}_{eh}\hat{C}^{hc}{}_{gx} \equiv \delta^y_x \ \hat{C}^{ab}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fg}{}_{eh}\hat{C}^{hc}{}_{ga}/4 \ , \tag{8c}$$

$$\text{(c)} \qquad \hat{C}^{yb}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fgc}{}_{h}\hat{C}^{h}{}_{egx} \equiv \delta^y_x \ \hat{C}^{ab}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fgc}{}_{h}\hat{C}^{h}{}_{ega}/4 \ . \tag{8d}$$

It is obvious from the constructions that these identities for four/five dimensions are valid in four/five *and lower* dimensions. So, for instance (6b) is also valid in four dimensions, but the only non-trivial information is in its trace in four dimensions, which is equivalent to (6a); on the otherhand, in five dimensions, the trace of the left hand side of (6b) is identically zero, giving a trivial result. Of course no Weyl candidates exist in dimensions less than four, but for the other tensors, these lemmas are nontrivial in lower dimensions. However, we shall not be concerned with dimensions less than four in this paper. Although stated for

[†] A 'candidate' of a tensor such as the Riemann or Weyl tensor is a tensor with the same index and trace properties, but sharing no other properties, such as differential properties; we shall designate such candidates with the symbol ˆ. So, for example a 'Weyl candidate tensor' $\hat{C}_{abcd}$ is defined by the properties $\hat{C}_{abcd} = \hat{C}_{[ab]cd} = \hat{C}_{ab[cd]}$, $\hat{C}_{a[bcd]} = 0$, $\hat{C}^a{}_{bca} = 0$. We shall follow the usual notation [2] with $R_{abcd}, C_{abcd}, S_{ab}, R$ for Riemann, Weyl, trace-free Ricci tensors and Ricci scalar respectively; their 'candidates' will be respectively $\hat{R}_{abcd}, \hat{C}_{abcd}, \hat{S}_{ab}, \hat{R}$. We shall also follow the usual notation for the Weyl and Ricci spinors respectively $\Psi_{ABCD}, \Phi_{ABC'D'}$, and scalar $\Lambda(= R/24)$; their 'candidates' will be respectively, $\hat{\Psi}_{ABCD}, \hat{\Phi}_{ABC'D'}, \hat{\Lambda}$. In addition we will use the Lanczos spinor $L_{ABCD'} = L_{(ABC)D'}$, and Lanczos tensor $L_{abc} = L_{[ab]c}$, $L^c{}_{ac} = 0 = L_{[abc]}$ [9,11] with corresponding 'candidates' $\hat{L}_{ABCD}$ and $\hat{L}_{abc}$.

Lanczos and Weyl candidates (which is all we require in this paper), a number of these results are valid for more general tensors; in particular the antisymmetry property $\hat{L}_{[abc]} = 0$ can be relaxed in Lemma 2, and the antisymmetry $\hat{C}_{a[bcd]} = 0$ can be relaxed in Lemma 3 and Lemma 4(a,b).

2. Simple Spinor Identities.
We begin with the following spinor result which generalises Penrose's original derivation [2] for Bel-Robinson tensors:
Theorem 1. A spinor which factorises according to $\top_{\mathcal{A}X\mathcal{A}'X'} = 4V_{\mathcal{S}X}\bar{V}_{\mathcal{S}'X'}$ satisfies

$$\top_{\mathcal{S}X\mathcal{S}'X'}\top^{\mathcal{S}Y\mathcal{S}'Y'} = \epsilon_X{}^Y\epsilon_{X'}{}^{Y'}\top_{\mathcal{S}A\mathcal{S}'A'}\top^{\mathcal{S}A\mathcal{S}'A'}/4 \tag{9}$$

where $\mathcal{S}, \mathcal{S}'$ each represent an *odd* number of spinor indices.
Proof.

$$V_{\mathcal{S}X}V^{\mathcal{S}Y} = V_{\mathcal{S}}{}^Y V^{\mathcal{S}}{}_X + \epsilon_X{}^Y V_{\mathcal{S}A}V^{\mathcal{S}A} = -V^{\mathcal{S}Y}V_{\mathcal{S}X} + \epsilon_X{}^Y V_{\mathcal{S}A}V^{\mathcal{S}A}$$

with the negative sign arising by 'see-sawing' the odd number of indices in $\mathcal{S}$. Hence

$$V_{\mathcal{S}X}V^{\mathcal{S}Y} = \epsilon_X{}^Y V_{\mathcal{S}A}V^{\mathcal{S}A}/2 \tag{10}$$

Multiplying by the complex conjugate

$$\bar{V}_{\mathcal{S}'X'}\bar{V}^{\mathcal{S}'Y'}V_{\mathcal{S}X}V^{\mathcal{S}Y} = \epsilon_X{}^Y V_{\mathcal{S}A}V^{\mathcal{S}A}\epsilon_{X'}{}^{Y'}\bar{V}_{\mathcal{S}'A'}\bar{V}^{\mathcal{S}'A'}/4$$

and substituting for $\top_{\mathcal{S}X\mathcal{S}'X'}$, $\top_{\mathcal{S}Y\mathcal{S}'Y'}$ gives the result. $\diamond$
From Theorem 1 we see that the types of indices in the collection of indices represented by $\mathcal{S}$ do not matter; only the fact that there is an odd number. In this paper we shall be concentrating on 4-index tensors $\mathcal{T}_{abcd}$ equivalently $\mathcal{T}_{ABCDA'B'C'D'}$; and in particular from [4],
• the superenergy spinor of the Weyl (candidate) spinor $\hat{\Psi}_{ABCD}$ (i.e., the Bel-Robinson superenergy spinor) is given by

$$\mathcal{T}[\hat{\Psi}]_{ABCDA'B'C'D'} = 4\hat{\Psi}_{ABCD}\bar{\hat{\Psi}}_{A'B'C'D'} \tag{11a}$$

• the superenergy spinor of the Ricci (candidate) spinor $\hat{\Phi}_{ABC'D'}$ is given by

$$\mathcal{T}[\hat{\Phi}]_{ABCDA'B'C'D'} = 4\hat{\Phi}_{ABC'D'}\bar{\hat{\Phi}}_{CDA'B'} \tag{11b}$$

• the superenergy spinor of the Ricci (candidate) scalar $\hat{\Lambda}$ is given by

$$\mathcal{T}[\hat{\Lambda}]_{ABCDA'B'C'D'} = 4\hat{\Lambda}^2(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{A'D'}\epsilon_{B'C'}) \tag{11c}$$

• the superenergy spinor of the Weyl-Ricci scalar (candidate) spinor $\hat{\chi}_{ABCD}$ is given by

$$\begin{aligned}
\mathcal{T}[\hat{\chi}]_{ABCDA'B'C'D'} &= 4\hat{\chi}_{ABCD}\bar{\hat{\chi}}_{A'B'C'D'} \\
&= 4\Big(\hat{\Psi}_{ABCD} + \hat{\Lambda}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})\Big)\Big(\bar{\hat{\Psi}}_{A'B'C'D'} + \hat{\Lambda}(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{A'D'}\epsilon_{B'C'})\Big) \\
&= \mathcal{T}[\hat{\Psi}] + \mathcal{T}[\hat{\Lambda}] \\
&\quad + 4\hat{\Lambda}\Big(\bar{\hat{\Psi}}_{A'B'C'D'}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}) + \hat{\Psi}_{ABCD}(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{A'D'}\epsilon_{B'C'})\Big) \, .
\end{aligned} \tag{11d}$$

Theorem 1 immediately specialises to,
Theorem 2. The four superenergy spinors in (11) $\mathcal{T}[\hat{\Psi}]$, $\mathcal{T}[\hat{\Phi}]$, $\mathcal{T}[\hat{\Lambda}]$, $\mathcal{T}[\hat{\chi}]$ all obey the identity

$$\mathcal{T}_{ABCXA'B'C'X'}\mathcal{T}^{ABCYA'B'C'Y'} = \epsilon_X{}^Y\epsilon_{X'}{}^{Y'}\mathcal{T}_{ABCDA'B'C'D'}\mathcal{T}^{ABCDA'B'C'D'}/4 \, . \tag{12}$$

$\diamond$

The simplicity of the above theorems is due to the fact that the superenergy spinors were simple direct products involving a spinor times its conjugate; it should be noted that even more general identities could have been obtained for these four spinors, as well as for more general spinors with the same simple product structure — not just from the point of view of relaxing the index symmetries, but also from freeing more indices, [2].

However the Bel superenergy spinor (the superenergy spinor for the Riemann (candidate) spinor) [2,4] and the Lanczos superenergy spinor (the superenergy spinor for the Lanczos (candidate) spinor), [4], do not have such a simple structure as can be seen below:

$$
\begin{aligned}
\mathcal{B}_{ABCDA'B'C'D'} &= 4(\hat{\chi}_{ABCD}\bar{\hat{\chi}}_{A'B'C'D'} + \hat{\Phi}_{ABC'D'}\bar{\hat{\Phi}}_{CDA'B'}) \\
&= 4\Big(\hat{\Psi}_{ABCD} + \hat{\Lambda}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})\Big) \\
&\quad \times \Big(\bar{\hat{\Psi}}_{A'B'C'D'} + \hat{\Lambda}(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{A'D'}\epsilon_{B'C'})\Big) + 4\hat{\Phi}_{ABC'D'}\bar{\hat{\Phi}}_{CDA'B'} \\
&= 4\Big(\hat{\Psi}_{ABCD}\bar{\hat{\Psi}}_{A'B'C'D'} + \hat{\Phi}_{ABC'D'}\bar{\hat{\Phi}}_{CDA'B'} \\
&\quad + \hat{\Lambda}^2(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{A'D'}\epsilon_{B'C'}) \\
&\quad + \hat{\Lambda}\hat{\Psi}_{ABCD}(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{A'D'}\epsilon_{B'C'}) + \hat{\Lambda}\bar{\hat{\Psi}}_{A'B'C'D'}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})\Big)
\end{aligned}
\tag{13}
$$

(where we have used the notation $\mathcal{B}$ rather than the more consistent $\mathcal{T}[\hat{R}]$ simply for ease of presentation), and

$$
\mathcal{T}[\hat{L}]_{ABCDA'B'C'D'} = 2\Big(\hat{L}_{ABCD'}\bar{\hat{L}}_{A'B'C'D} + \hat{L}_{ABDC'}\bar{\hat{L}}_{A'B'D'C}\Big)
\tag{14}
$$

where $\hat{L}_{ABCD'} = \hat{L}_{(ABC)D'}$.

It is easy to see that simple identities such as (12) do not hold in these cases. However this does not rule out the possibility of other, more complicated, identities. We shall look further at both of these tensors in Section 6, and at possible identities for the Bel tensor in Section 4.

3. Simple Tensor Identities.

We now wish to confirm the tensor versions of the four identities in Theorem 2 by tensor means. We shall discover that although the above four identities had essentially the same spinor proofs, the proofs for their superenergy tensor counterparts require very different amounts of calculations. We first give the corresponding n-dimensional basic superenergy tensors of the appropriate double 2-forms.

In n-dimensional spaces from [3],

• the basic Bel-Robinson tensor [1] (equivalent to the Bel-Robinson spinor $\mathcal{T}[\hat{\Psi}]$) is given by

$$
\mathcal{T}[\hat{C}]_{abcd} = \hat{C}_{apcq}\hat{C}_b{}^p{}_d{}^q + \hat{C}_{apdq}\hat{C}_b{}^p{}_c{}^q - \frac{1}{2}g_{ab}\hat{C}_{rpcq}\hat{C}^{rp}{}_d{}^q - \frac{1}{2}g_{cd}\hat{C}_{aprq}\hat{C}_b{}^{prq} + \frac{1}{8}g_{ab}g_{cd}\hat{C}_{sprq}\hat{C}^{sprq} , \tag{15}
$$

• the basic trace-free Ricci superenergy tensor (equivalent to the superenergy spinor for the Ricci (candidate) spinor $\mathcal{T}[\hat{\Phi}]$) is given — via the tensor[†] $\hat{E}_{abcd} = (\hat{S}_{ac}g_{bd} - \hat{S}_{ad}g_{bc} + \hat{S}_{bd}g_{ac} - \hat{S}_{bc}g_{ad})/(n-2)$ — by,

$$
\begin{aligned}
\mathcal{T}[\hat{E}]_{abcd} &= \hat{E}_{aecf}\hat{E}_b{}^e{}_d{}^f + \hat{E}_{aedf}\hat{E}_b{}^e{}_c{}^f - \frac{1}{2}g_{ab}\hat{E}_{efcg}\hat{E}^{ef}{}_d{}^g - \frac{1}{2}g_{cd}\hat{E}_{aefg}\hat{E}_b{}^{efg} + \frac{1}{8}g_{ab}g_{cd}\hat{E}_{efgh}\hat{E}_{efgh} \\
&= 4\Big(\hat{S}_{ab}\hat{S}_{cd} + \frac{n-4}{2}\hat{S}_{a(c}\hat{S}_{d)b} - \hat{S}_{bp}\hat{S}_{(d}{}^p g_{c)a} - \hat{S}_{ap}\hat{S}_{(d}{}^p g_{c)b} + \frac{6-n}{4}\hat{S}_{cp}\hat{S}_d{}^p g_{ab} + \frac{6-n}{4}\hat{S}_{ap}\hat{S}_b{}^p g_{cd} \\
&\quad + \frac{n-6}{8}g_{ab}g_{cd}\hat{S}_{pq}\hat{S}^{pq} + \frac{1}{2}g_{a(c}g_{d)b}\hat{S}_{pq}\hat{S}^{pq}\Big)/(n-2)^2 ,
\end{aligned}
\tag{16}
$$

[†] Note that the superenergy tensor constructed for the trace-free Ricci candidate tensor $\hat{S}_{ab}$ via the double 2-form $\hat{E}_{abcd}$ is different from the superenergy tensor constructed for the trace-free Ricci candidate tensor directly via the double 1-form $\hat{S}_{ab}$ [3].

• the basic Ricci scalar superenergy tensor (equivalent to the superenergy spinor for the Ricci (candidate) scalar $\mathcal{T}[\hat{\Lambda}]$) is given — via the tensor $\hat{G}_{abcd} = \hat{R}(g_{ac}g_{bd} - g_{ad}g_{bc})/n(n-1)$ — by

$$
\begin{aligned}
\mathcal{T}[\hat{\Lambda}]_{abcd} &= \hat{G}_{aecf}\hat{G}_b{}^e{}_d{}^f + \hat{G}_{aedf}\hat{G}_b{}^e{}_c{}^f - \frac{1}{2}g_{ab}\hat{G}_{efcg}\hat{G}^{ef}{}_d{}^g - \frac{1}{2}g_{cd}\hat{G}_{aefg}\hat{G}_b{}^{efg} + \frac{1}{8}g_{ab}g_{cd}\hat{G}_{efgh}\hat{G}^{efgh} \\
&= \hat{R}^2\Big(2(n-2)g_{a(c}g_{d)b} + \frac{n^2 - 9n + 16}{4}g_{ab}g_{cd}\Big)/n^2(n-1)^2 \ ,
\end{aligned}
\tag{17}
$$

• the basic superenergy tensor for the $\hat{\chi}$ (candidate) tensor (equivalent to the superenergy spinor for the $\hat{\chi}$ (candidate) spinor $\mathcal{T}[\hat{\chi}]$) is given — via the tensor $\hat{\chi}_{abcd} = \hat{C}_{abcd} + \hat{R}(g_{ac}g_{bd} - g_{ad}g_{bc})/n(n-1)$ — by

$$
\begin{aligned}
\mathcal{T}[\hat{\chi}]_{abcd} &= \hat{\chi}_{aecf}\hat{\chi}_b{}^e{}_d{}^f + \hat{\chi}_{aedf}\hat{\chi}_b{}^e{}_c{}^f - \frac{1}{2}g_{ab}\hat{\chi}_{efcg}\hat{\chi}^{ef}{}_d{}^g - \frac{1}{2}g_{cd}\hat{\chi}_{aefg}\hat{\chi}_b{}^{efg} + \frac{1}{8}g_{ab}g_{cd}\hat{\chi}_{efgh}\hat{\chi}^{efgh} \\
&= \mathcal{T}[\hat{C}]_{abcd} + \mathcal{T}[\hat{\Lambda}]_{abcd} + 2\hat{R}\big(\hat{C}_{acbd} + \hat{C}_{adbc}\big)/n(n-1) \ .
\end{aligned}
\tag{18}
$$

In 4-dimensional space [3, 26],
• $\mathcal{T}[\hat{C}]_{abcd}$ has the same form as the n-dimensional case
• $\mathcal{T}[\hat{E}]_{abcd}$ simplifies to

$$
\mathcal{T}[\hat{E}]_{abcd} = \hat{S}_{ab}\hat{S}_{cd} + \hat{S}_{ap}\hat{S}_b{}^p g_{cd} + \hat{S}_{cp}\hat{S}_d{}^p g_{ab} - 3\hat{S}_{p(a}\hat{S}_{b}{}^p g_{cd)} + \frac{1}{4}\hat{S}_{pq}\hat{S}^{pq}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}) \ , \tag{19}
$$

• $\mathcal{T}[\hat{\Lambda}]_{abcd}$ simplifies to

$$
\mathcal{T}[\hat{\Lambda}]_{abcd} = \hat{R}^2(4g_{a(c}g_{d)b} - g_{ab}g_{cd})/144 \ , \tag{20}
$$

• $\mathcal{T}[\hat{\chi}]_{abcd}$ simplifies to

$$
\mathcal{T}[\hat{\chi}]_{abcd} = \mathcal{T}[\hat{C}]_{abcd} + \frac{\hat{R}}{6}(\hat{C}_{acbd} + \hat{C}_{adbc}) + \mathcal{T}[\hat{\Lambda}]_{abcd} \ . \tag{21}
$$

It is clear that all of the above superenergy tensors have the properties

$$
\mathcal{T}_{abcd} = \mathcal{T}_{(ab)(cd)} \tag{22}
$$

and some have additional symmetry properties, e.g., for the Bel tensor (in four and five dimensions), and the Lanczos superenergy tensor (in four dimensions), $\mathcal{T}_{abcd} = \mathcal{T}_{(abcd)}$ as we shall show in Section 6. All of the above are labelled *basic* superenergy tensors to distinguish from the more *general* superenergy tensors which can be obtained by taking linear combinations — with positive constant coefficients — of different basic superenergy tensors, obtained by index permutations [3].

We now give the 4-dimensional tensor counterparts of Theorem 2.

Theorem 2a. In 4-dimensional spaces the Bel-Robinson tensor $\mathcal{T}[C]_{abcd}$ in (15) satisfies

$$
\mathcal{T}[\hat{C}]_{abcx}\mathcal{T}[\hat{C}]^{abcy} = \delta_x^y \ \mathcal{T}[\hat{C}]_{abcd}\mathcal{T}[\hat{C}]^{abcd}/4 \tag{23}
$$

Proof. Substituting directly we obtain

$$
\begin{aligned}
\mathcal{T}[\hat{C}]&_{abcx}\mathcal{T}[\hat{C}]^{abcy} - \frac{1}{4}\delta_x^y \ \mathcal{T}[\hat{C}]_{abcd}\mathcal{T}[\hat{C}]^{abcd} \\
&= 2\hat{C}^{yb}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fg}{}_{eh}\hat{C}^{hc}{}_{gx} + 2\hat{C}^{yb}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fgc}{}_h\hat{C}^h{}_{egx} - 2\hat{C}^{ab}{}_{cd}\hat{C}^{cd}{}_{eb}\hat{C}^{e}{}_{gh}{}^y\hat{C}_a{}^{gh}{}_y \\
&\quad - \hat{C}^{yb}{}_{cd}\hat{C}^{cd}{}_{eb}\hat{C}^{ef}{}_{gh}\hat{C}^{gh}{}_{xf} + \hat{C}_{abcd}\hat{C}^{abcd}C^{ey}{}_{gh}\hat{C}^{gh}{}_{ex} \\
&\quad + \frac{1}{4}\delta_x^y\hat{C}^{ab}{}_{cd}\hat{C}^{cd}{}_{eb}\hat{C}^{ef}{}_{gh}\hat{C}^{gh}{}_{af} - \frac{1}{16}\delta_x^y\hat{C}^{abcd}\hat{C}_{abcd}\hat{C}^{efgh}\hat{C}_{efgh} \\
&\quad - \frac{1}{4}\delta_x^y\Big(2\hat{C}^{ab}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fg}{}_{eh}\hat{C}^{hc}{}_{ga} + 2\hat{C}^{ab}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fgc}{}_h\hat{C}^h{}_{ega} \\
&\qquad\qquad - 2\hat{C}^{ab}{}_{cd}\hat{C}^{cd}{}_{eb}\hat{C}^{ef}{}_{gh}\hat{C}^{gh}{}_{af} + \frac{1}{16}\hat{C}^{abcd}\hat{C}_{abcd}\hat{C}^{efgh}\hat{C}_{efgh}\Big) \ .
\end{aligned}
\tag{24}
$$

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Using Lemma 4a a number of times gives the simpler expression

$$
\mathcal{T}[\hat{C}]_{abcx}\mathcal{T}[\hat{C}]^{abcy} - \frac{1}{4}\delta^y_x \ \mathcal{T}[\hat{C}]_{abcd}\mathcal{T}[\hat{C}]^{abcd}
$$

$$
= 2\hat{C}^{yb}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fg}{}_{eh}\hat{C}^{hc}{}_{gx} + 2\hat{C}^{yb}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fgc}{}_{h}\hat{C}^{h}{}_{egx} \tag{25}
$$

$$
- \frac{1}{4}\delta^y_x\left(2\hat{C}^{ab}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fg}{}_{eh}\hat{C}^{hc}{}_{ga} + 2\hat{C}^{ab}{}_{cd}\hat{C}^{de}{}_{bf}\hat{C}^{fgc}{}_{h}\hat{C}^{h}{}_{ega}\right) .
$$

We can now apply Lemma 4b to the first and third terms, and Lemma 4c to the second and fourth terms, to obtain the required result. $\diamond$

Theorem 2b.
In 4-dimensional spaces, the superenergy tensor $\mathcal{T}[\hat{E}]_{abcd}$ given in (19) satisfies

$$
\mathcal{T}[\hat{E}]_{abcx}\mathcal{T}[\hat{E}]^{abcy} = \delta^y_x \ \mathcal{T}[\hat{E}]_{abcd}\mathcal{T}[\hat{E}]^{abcd}/4 \tag{26}
$$

Proof.
Substituting directly we obtain

$$
\mathcal{T}[\hat{E}]_{abcx}\mathcal{T}[\hat{E}]^{abcy} - \frac{1}{4}\delta^y_x \ \mathcal{T}[\hat{E}]_{abcd}\mathcal{T}[\hat{E}]^{abcd}
$$

$$
= -3\hat{S}^y{}_b\hat{S}^b{}_c\hat{S}^c{}_d\hat{S}^d{}_x + \frac{3}{2}\hat{S}^{yb}\hat{S}_{xb}\hat{S}_{cd}\hat{S}^{cd} + \hat{S}^y{}_x\hat{S}^a{}_b\hat{S}^b{}_c\hat{S}^c{}_a + \frac{3}{4}\delta^y_x\hat{S}^a{}_b\hat{S}^b{}_c\hat{S}^c{}_d\hat{S}^d{}_a - \frac{3}{8}\delta^y_x\hat{S}^{ab}\hat{S}_{ab}\hat{S}_{cd}\hat{S}^{cd} . \tag{27}
$$

But the right-hand side of this equation is precisely

$$
\delta^y_{[x} \ \hat{S}^a{}_a\hat{S}^b{}_b\hat{S}^c{}_c\hat{S}^d{}_{d]} \equiv 0, \tag{28}
$$

where we have made use of Lemma 1. $\diamond$

Theorem 2c.
In 4-dimensional spaces, the superenergy tensor $\mathcal{T}[\hat{\Lambda}]_{abcd}$ given in (20) satisfies

$$
\mathcal{T}[\hat{\Lambda}]_{abcx}\mathcal{T}[\hat{\Lambda}]^{abcy} = \delta^y_x \ \mathcal{T}[\hat{\Lambda}]_{abcd}\mathcal{T}[\hat{\Lambda}]^{abcd}/4 \tag{29}
$$

Proof. The result follows from a direct calculation. $\diamond$

Theorem 2d.
In 4-dimensional spaces, the superenergy tensor $\mathcal{T}[\hat{\chi}]_{abcd}$ given in (21) satisfies

$$
\mathcal{T}[\hat{\chi}]_{abcx}\mathcal{T}[\hat{\chi}]^{abcy} = \delta^y_x \ \mathcal{T}[\hat{\chi}]_{abcd}\mathcal{T}[\hat{\chi}]^{abcd}/4 \tag{30}
$$

Proof.

$$
\mathcal{T}[\hat{\chi}]_{abcx}\mathcal{T}[\hat{\chi}]^{abcy}
$$

$$
= \left(\mathcal{T}[\hat{C}]_{abcx} + \mathcal{T}[\hat{\Lambda}]_{abcx} + \frac{\hat{R}}{6}(\hat{C}_{acbx} + \hat{C}_{axbc})\right)\left(\mathcal{T}[\hat{C}]^{abcy} + \mathcal{T}[\hat{\Lambda}]^{abcy} + \frac{\hat{R}}{6}(\hat{C}^{acby} + \hat{C}^{aybc})\right)
$$

$$
= \mathcal{T}[\hat{C}]_{abcx}\mathcal{T}[\hat{C}]^{abcy} + \mathcal{T}[\hat{\Lambda}]_{abcx}\mathcal{T}[\hat{\Lambda}]^{abcy} + \frac{\hat{R}}{6}\left(\mathcal{T}[\hat{C}]^{abcy}(\hat{C}_{acbx} + \hat{C}_{axbc}) + \mathcal{T}[\hat{C}]_{abcx}(\hat{C}^{acby} + \hat{C}^{aybc})\right)
$$

$$
+ \frac{\hat{R}^2}{144}\hat{C}_{abcd}\hat{C}^{abcd}\delta^y_x \tag{31}
$$

where the last term was obtained using Lemma 4a.
We can apply Theorems 2a and 2c to the first and second terms respectively; however, to complete the proof we need to use *cubic* identities for the Weyl tensor; the details are given in [23]. $\diamond$

We note that in Theorems 2,a,b,d explicit 4-dimensional fddis were used in the proofs; it seems unlikely that *direct* generalisations can be obtained by using analogous fddis in higher dimensions; more likely, higher order identities would need to be considered. Theorem 2c was obtained by a direct calculation, and in fact an analogous identity is clearly obtainable in n dimensions because of the very simple structure of the superenergy tensor in this case.

4. Absence of simple identities for the Bel Superenergy Tensor.

The Bel tensor, the superenergy tensor for the Riemann tensor, in n dimensions [3] is

$$\mathcal{B}_{abcd} = \hat{R}_{apcq}\hat{R}_b{}^p{}_d{}^q + \hat{R}_{apdq}\hat{R}_b{}^p{}_c{}^q - \frac{1}{2}g_{ab}\hat{R}_{rpcq}\hat{R}^{rp}{}_d{}^q - \frac{1}{2}g_{cd}\hat{R}_{aprq}\hat{R}_b{}^{prq} + \frac{1}{8}g_{ab}g_{cd}\hat{R}_{sprq}\hat{R}^{sprq} \quad (32)$$

with the obvious properties

$$\mathcal{B}_{abcd} = \mathcal{B}_{(ab)cd} = \mathcal{B}_{ab(cd)} = \mathcal{B}_{cdab}, \qquad \mathcal{B}^a{}_{acd} = 0, \quad (33)$$

(We continue to use the notation $\mathcal{B}_{abcd}$ rather than the more consistent $\mathcal{T}[\hat{R}]_{abcd}$.)
Substituting the usual decomposition of the Riemann tensor gives the alternative form [3], [15]

$$\mathcal{B}_{abcd} = \mathcal{T}[\hat{C}]_{abcd} + \mathcal{T}[\hat{E}]_{abcd} + \mathcal{Q}_{abcd} \quad (32')$$

where

$$\mathcal{Q}_{abcd} = \frac{1}{n-2}\left(-4\hat{C}^i{}_{(cd)(a}\hat{S}_{b)i} - 4\hat{C}^i{}_{(ab)(c}\hat{S}_{d)i} + 2\hat{S}_{ij}\left(\hat{C}_a{}^j{}_{(c}{}^i g_{d)b} - \hat{C}_c{}^j{}_d{}^i g_{ab} + \hat{C}_b{}^j{}_{(c}{}^i g_{d)a} - \hat{C}_a{}^j{}_b{}^i g_{cd}\right)\right)$$
$$+ \frac{2\hat{R}}{n(n-1)}(\hat{C}_{acbd} + \hat{C}_{adbc}) . \quad (34)$$

In four dimensions we get

$$\mathcal{Q}_{abcd} = \frac{\hat{R}}{6}(\hat{C}_{acbd} + \hat{C}_{adbc}) .$$

This last simplification is not obvious in tensors, although it is in spinors (11d); by tensors, it is obtained either by manipulation with duals [3], or via a 4-dimensional ddi, [15]

$$\hat{S}^e{}_f \hat{C}_{[ab}{}^{[cd}\delta_{e]}^{f]} \equiv 0 . \quad (35)$$

The Bel tensor is a generalisation of the Bel-Robinson tensor, and an obvious question is whether it also satisfies similar types of quadratic identity in four dimensions as the Bel-Robinson tensor does. From spinor considerations it does not look very hopeful, so we investigate the possibility via examples rather than look for general results.

Because of the additional terms in the Bel tensor compared to the Bel-Robinson tensor we introduce

$$\mathcal{B}_a{}^c{}_{cb} = \mathcal{B}_{ab} = \mathcal{B}_{ba}, \qquad \mathcal{B} = \mathcal{B}^a{}_a \quad (36)$$

and so we consider the general quadratic identity with two free indices with the structure

$$k_1\mathcal{B}_{abcx}\mathcal{B}^{abcy} + k_2\mathcal{B}_{abcx}\mathcal{B}^{acby} + k_3\mathcal{B}_{abx}{}^y\mathcal{B}^{ab} + k_4\mathcal{B}_{axb}{}^y\mathcal{B}^{ab} + k_5\mathcal{B}_{ax}\mathcal{B}^{ay} + k_6\mathcal{B}\mathcal{B}_x{}^y \propto \delta_x^y \quad (37)$$

with constants $k_1, ..., k_6$. By substituting the Bel tensors of explicit spaces[†] in the left hand side, we are led to conjecture that, in 4-dimensional spaces, the Bel tensor satisfies the quadratic identity

$$2\mathcal{B}_{a[bc]x}\mathcal{B}^{a[bc]y} - \mathcal{B}_{xa}\mathcal{B}^{ay} + 2\mathcal{B}_{ab}\mathcal{B}_a{}^{[by]}{}_x + \frac{1}{2}\mathcal{B}\mathcal{B}_x{}^y = \delta_x^y\left(2\mathcal{B}_{a[bc]d}\mathcal{B}^{a[bc]d} - 2\mathcal{B}_{ab}\mathcal{B}^{ab} + \frac{1}{2}\mathcal{B}^2\right)/4 . \quad (38)$$

[†] In fact the choice of the van Stockum metric [24] $ds^2 = -dt^2 - 2a\rho^2 dtd\phi - e^{-2a\rho}d\rho^2 + e^{-2a\rho}dz^2 + (\rho^2 - a^2\rho^4)d\phi^2$ (available in *GRTensorII* [25]) will give a one parameter solution for the constants used above, and substituting these values leads to the identity (38).

So it appears that the Bel tensor *may* have a quadratic identity, which rather surprisingly does not reduce to the Bel-Robinson identity in the vacuum case. However, closer inspection reveals that this identity is *trivial* in the following sense: the properties $\mathcal{B}^a{}_{[bc]}{}^d = \mathcal{B}^{[a}{}_{[bc]}{}^{d]} = \mathcal{B}_{[b}{}^{[ad]}{}_{c]}$, $\mathcal{B}_{a[bcd]} = 0$ mean that we can consider $B_{abcd} \equiv \mathcal{B}^{[c}{}_{[ab]}{}^{d]}$ as a Riemann candidate and (38) becomes

$$\tilde{B}_{abcx}\tilde{B}^{abcy} = \delta_x^y\ \tilde{B}_{abcd}\tilde{B}^{abcd}/4 \tag{39}$$

where $\tilde{B}_{abcd}$ is the trace-free part of B_{abcd}, i.e., its Weyl candidate. But (39) is just the identity in Lemma 4a which is a consequence of *only* the trace-free 2-form structure and the fact that we are in 4-dimensional space; *it has nothing to do with the superenergy structure of $\mathcal{B}_{abcd}$ as a linear combination of products of Riemann candidates.*

Hence this identity is of no interest to us in the context of superenergy tensors, and so,

Theorem 3. In 4-dimensional spaces, the Bel superenergy tensor (32) $\mathcal{B}_{abcd}$, for a Riemann (candidate) tensor $\hat{R}_{abcd}$, does not satisfy any non-trivial quadratic identity with the structure (37). $\diamondsuit$

5. Absence of simple identities for Bel-Robinson Superenergy tensor in higher dimensions.

As noted above, it has been found [3,15] that the Bel-Robinson tensor is completely symmetric in *five* (and lower) dimensions. This raises the question as to whether the above quadratic identity for the Bel-Robinson tensor is also valid in five dimensions; or, more generally, whether there exists *any* quadratic identity with two free indices for the Bel-Robinson tensor in *five* dimensions.

Since in five dimensions the Bel-Robinson tensor is still fully symmetric, but not trace-free, the most general quadratic identity with two free indices which could exist would have to have the structure,

$$k_1\mathcal{T}_{abcx}\mathcal{T}^{abcy} + k_2\mathcal{T}_{abx}{}^y\mathcal{T}^{ab} + k_3\mathcal{T}_{ax}\mathcal{T}^{ay} + k_4\mathcal{T}\,\mathcal{T}_x^y \propto\ \delta_x^y \tag{40}$$

for constants k_1, k_2, k_3, k_4, where

$$\mathcal{T}_{abcd} = \mathcal{T}_{(abcd)}, \qquad \mathcal{T}_{abc}{}^c = \mathcal{T}_{ab} = \mathcal{T}_{ba}, \qquad \mathcal{T}_a{}^a = \mathcal{T}\ . \tag{41}$$

To try to retrace the complicated tensor calculations of Theorem 2a, and try to replace the 4-dimensional fddis used there with higher dimensional fddis would be a very complicated procedure; so we try first to obtain a simple counterexample, and we easily obtain the following negative result,

Theorem 4. In 5-dimensional spaces, the Bel-Robinson tensor $\mathcal{T}_{abcd}$ does not satisfy any non-trivial quadratic identity of the form (40).

Proof. Generalising the 4-dimensional Kerr metric g_{ab}^K to five dimensions as $ds^2 = g_{ab}^K dx^a dx^b + dx_5^2$, we can calculate the Bel-Robinson tensor explicitly, and when we substitute it into the left hand side of the above expression (40), we obtain

$$k_1\mathcal{T}_{abcx}\mathcal{T}^{abcy} + k_2\mathcal{T}_{abx}{}^y\mathcal{T}^{ab} + k_3\mathcal{T}_{ax}\mathcal{T}^{ay} + k_4\mathcal{T}\,\mathcal{T}_x^y = K\delta_x^y + (J-K)\delta_x^5\delta_5^y \tag{42}$$

where

$$K = 144M^4k_1/(x^2 + a^2y^2)^6,\ \ J = -36M^4(x^6 - 15a^2y^2x^4 + 15a^4y^4x^2 - a^6y^6)^2\big(k_1 + k_2 + k_3 + k_4\big)$$

in Boyer-Lindquist coordinates. So clearly, in general, there are no choices of the constants k_1, k_2, k_3, k_4 which will give us a non-trivial identity. $\diamondsuit$

Note that we have used Bel-Robinson tensors constructed from Weyl tensors and not the more general candidates in this proof. This not only gives a stronger result than if candidates had been used, but was obtained very simply using *GRTensorII* [25].

In spaces of dimension $n > 5$ the Bel-Robinson tensor is no longer completely symmetric and so to investigate the most general possible quadratic identity with two free indices we would need to consider a much more complicated form than in (40).

6. Index symmetry of Bel-Robinson and Lanczos superenergy tensors.

In this section we will determine the kernel fddi for two results in four dimensions; we will then show by considering the analogous higher dimensional fddis how, in one case there is a simple generalisation to five (and only five) dimensions, while in the other there is no direct generalisation to higher dimensions.

The Bel-Robinson spinor (11a) is trivially symmetric in all indices. On the other hand, the only obvious symmetries from the Bel-Robinson tensor (15) are $\mathcal{T}_{abcd} = \mathcal{T}_{(ab)cd} = \mathcal{T}_{ab(cd)} = \mathcal{T}_{cdab}$. To check if it is fully symmetric in *all* indices we examine

$$\mathcal{T}[\hat{C}]_{a[bc]d} = \frac{1}{4}\hat{C}_{adef}\hat{C}_{bc}{}^{ef} - \hat{C}_{eaf[b}\hat{C}_{c]}{}^e{}_d{}^f - \hat{C}_{fge[a}g_{d][b}\hat{C}_{c]}{}^{efg} + \frac{1}{8}g_{a[b}g_{c]d}\hat{C}_{sprq}\hat{C}^{sprq} \ . \qquad (43)$$

We know from spinors that it must be symmetric in all indices in (at least) four dimensions, so the structure of (43) invites comparison with the 4-dimensional ddi for the Weyl tensor in Lemma 3 contracted with another Weyl tensor, i.e.,

$$0 \equiv \hat{C}_{[ib}{}^{[kl}\delta_{c]}^{a]}\hat{C}_{kl}{}^{id} \ , \qquad (44)$$

the right hand side of which when expanded coincides precisely with (43).

To determine whether the same result is valid in five dimensions, we consider the analogous *five* dimensional fddi in Lemma 3, and when we construct

$$0 \equiv \hat{C}_{[bc}{}^{[ad}\delta_{ij]}^{kl]}\hat{C}_{kl}{}^{ij} \ , \qquad (45)$$

we find that its expanded right hand side also coincides precisely with (43).

So we have demonstrated that the fact that the Bel-Robinson tensor (15) is fully symmetric in *five (and lower) dimensions*[†] can be seen as a simple consequence of one 5-dimensional fddi. (This result was originally obtained for four and five dimensions seperately using duals in [3], and subsequently by the present method in [15].)

For higher dimensions, from the viewpoint of fddis, we note that the next fddi $\hat{C}_{[ab}{}^{[fg}\delta_{cde]}^{hij]} = 0$ has too many indices to yield (43) by a contraction with one Weyl tensor. However, it is easy to show conclusively that $\mathcal{T}[\hat{C}]_{abcd}$ is not symmetric in higher dimensions by taking the double trace [3, 15].

The Lanczos superenergy spinor has the obvious symmetries $\mathcal{T}[\hat{L}]_{ABCDA'B'C'D'} = \mathcal{T}[\hat{L}]_{(AB)CD(A'B')C'D'} = \mathcal{T}[\hat{L}]_{AB(CD)A'B'(C'D')}$ and the more general Lanczos superenergy spinor

$$\tilde{\mathcal{T}}[\hat{L}]_{ABCDA'B'C'D'} = \left(\mathcal{T}[\hat{L}]_{ABCDA'B'C'D'} + \mathcal{T}[\hat{L}]_{CDABC'D'A'B'}\right) \qquad (46)$$

is clearly symmetric in all indices.

The basic Lanczos superenergy tensor (equivalent to the superenergy spinor for the Lanczos (candidate) spinor $\mathcal{T}[\hat{L}]_{ABCDA'B'C'D'}$) is given by [3, 15] in n dimensions

$$\mathcal{T}[\hat{L}]_{abcd} = \hat{L}_{aic}\hat{L}_b{}^i{}_d + \hat{L}_{aid}\hat{L}_b{}^i{}_c - \frac{1}{2}g_{ab}\hat{L}_{ijc}\hat{L}^{ij}{}_d - g_{cd}\hat{L}_{aij}\hat{L}_b{}^{ij} + \frac{1}{4}g_{ab}g_{cd}\hat{L}_{ijk}\hat{L}^{ijk} \ . \qquad (47)$$

[†] The *five* dimensional ddi (45)

$$\frac{1}{4}\hat{C}_{adef}\hat{C}_{bc}{}^{ef} - \hat{C}_{eaf[b}\hat{C}_{c]}{}^e{}_d{}^f - \hat{C}_{fge[a}g_{d][b}\hat{C}_{c]}{}^{efg} + \frac{1}{8}g_{a[b}g_{c]d}\hat{C}_{sprq}\hat{C}^{sprq} \equiv 0$$

which we have just exploited, is of course also valid in four dimensions; in four dimensions we can use Lemma 4a on the penultimate term and obtain the similar but simpler *four* dimensional ddi,

$$\frac{1}{4}\hat{C}_{adef}\hat{C}_{bc}{}^{ef} - \hat{C}_{eaf[b}\hat{C}_{c]}{}^e{}_d{}^f - \frac{1}{8}g_{a[b}g_{c]d}\hat{C}_{sprq}\hat{C}^{sprq} \equiv 0 \ .$$

which is just the identity (44). Deser [6] has pointed out the significance of this identity (44) in the symmetry structure of the Bel-Robinson tensor in four dimensions; here we also see the significance of the 5-dimensional counterpart (45) in the symmetry structure of the Bel-Robinson tensor in *five* dimensions.

It has the obvious properties $\mathcal{T}[\hat{L}]_{abcd} = \mathcal{T}[\hat{L}]_{(ab)(cd)}$, but not the block symmetry $\mathcal{T}_{abcd} = \mathcal{T}_{cdab}$. The more general superenergy tensor

$$\tilde{\mathcal{T}}[\hat{L}]_{abcd} = \left(\mathcal{T}[\hat{L}]_{abcd} + \mathcal{T}[\hat{L}]_{cdab} \right)/2 \ , \tag{48}$$

which is equivalent to (46), does not obviously appear to be completely symmetric, as we know it must be in four dimensions at least. To determine if $\tilde{\mathcal{T}}[\hat{L}]_{abcd}$ is symmetric over *all* indices we examine

$$\tilde{\mathcal{T}}[\hat{L}]^a{}_{[bc]}{}^d = \frac{1}{4}\hat{L}_{bce}\hat{L}^{ade} - \hat{L}^{[a}{}_{e[b}\hat{L}_{c]}{}^{|e|d} - \frac{1}{2}\delta^{[a}_{[b}\hat{L}_{|ef|c]}\hat{L}^{|ef|d]} - \delta^{[a}_{[b}\hat{L}_{c]ef}\hat{L}^{d]ef} + \frac{1}{4}\delta^a_{[b}\delta^d_{c]}\hat{L}_{ijk}\hat{L}^{ijk} \ . \tag{49}$$

The structure of (49) (including two deltas) invites comparison with the 4-dimensional ddi for the Lanczos tensor in Lemma 2 contracted with another Lanczos tensor, i.e.,

$$0 \equiv \hat{L}_{[ab}{}^{[e}\delta^{fg]}_{cd]}L^{cd}{}_g \tag{50}$$

the right hand side of which when expanded coincides precisely with (49). So we retrieve the result in [15],

Theorem 5. A Lanczos superenergy tensor $\tilde{\mathcal{T}}[\hat{L}]_{abcd} = \left(\mathcal{T}[\hat{L}]_{abcd} + \mathcal{T}[\hat{L}]_{cdab} \right)/2$ where $\mathcal{T}[\hat{L}]_{abcd}$ is given by (47) is symmetric in all indices in four dimensions. $\diamondsuit$

To determine whether the same proof is valid in five dimensions, we consider the analogous *five* dimensional fddi in Lemma 2, and we immediately see that there are too many free indices to yield (49) by a contraction with one Lanczos tensor.

7. Tensor derivation of new electromagnetic conservation law.

We now wish to look at a particular Lanczos candidate,

$$L_{abc} = F_{ab;c} \tag{51}$$

where F_{ab} is an electromagnetic field tensor which satisfies the source-free Maxwell's equations

$$F^a{}_{b;a} = 0 = F_{[ab;c]} \tag{52}$$

and so the properties $L^a{}_{ba} = 0 = L_{[abc]}$ of a Lanczos candidate are automatically satisfied. Hence we could choose the tensor (47) with the above substitution (51) and obtain

$$\mathcal{T}[\nabla F]_{abcd} = F_{ai;c}F_b{}^i{}_{;d} + F_{ai;d}F_b{}^i{}_{;c} - \frac{1}{2}g_{ab}F_{ij;c}F^{ij}{}_{;d} - g_{cd}F_{ai;j}F_b{}^{i;j} + \frac{1}{4}g_{ab}g_{cd}F_{ij;k}F^{ij;k} \tag{53}$$

as a superenergy tensor for the electromagnetic field. In fact Senovilla [3] has shown that a tensor $\mathcal{C}_{abcd}$, suggested by Chevreton [27] as an analogue for the Bel-Robinson tensor in an electromagnetic field, is just a linear combination of two such superenergy tensors

$$\mathcal{C}_{abcd} = \left(\mathcal{T}[\nabla F]_{abcd} + \mathcal{T}[\nabla F]_{cdab} \right)/2 \ . \tag{54}$$

This tensor was shown in [27] to have important properties *in flat space*: in particular it is divergence-free, but this property is not valid, in general, in curved spaces. Recently Bergqvist, Eriksson and Senovilla [5] have used the spinor equivalent of the Chevreton tensor and given two interesting properties *in curved space* for source-free Einstein-Maxwell fields:

• the Chevreton tensor is fully symmetric

• the trace of the Chevreton tensor is divergence free.

We now wish to consider the tensor versions of these results. The first of these properties is just a special case of the result previously derived in tensors for Lanczos candidates in 4-dimensional spaces [15], and given at the end of Section 4. The second property was deduced from the spinor form of the divergence of $\mathcal{C}_{abcd}$ and Bergqvist, Eriksson and Senovilla [5] remark that the proof of this result is far from obvious from

the tensor point of view. We shall now demonstrate that the result in [5] can be obtained in a direct and straightforward manner — until the complication at the last stages where *two fddis valid in four dimensions* have to be used explicitly.

Theorem 6. In four dimensions, the non-zero trace $\mathcal{C}_{ab} \equiv \mathcal{C}_{abc}{}^c$ of the Chevreton superenergy tensor $\mathcal{C}_{abcd}$ is symmetric, trace-free and divergence-free, i.e., $\mathcal{C}_a{}^b{}_{;b} = 0$.

Proof.

The non-zero trace of the Lanczos superenergy tensor is given by

$$\mathcal{C}_{ab} \equiv \mathcal{C}_{abc}{}^c = -L_{aef}L_b{}^{ef} + \frac{1}{4}g_{ab}L_{cef}L^{cef} \tag{55}$$

and it is clear that it is symmetric and trace-free. The divergence is

$$\mathcal{C}_a{}^b{}_{;b} = -L_{aef}L^{bef}{}_{;b} - L_{aef;b}L^{bef} + \frac{1}{2}L_{cef;a}L^{cef} \tag{56}$$

Now substituting (52) and simplifying gives

$$\begin{aligned}
2\mathcal{C}_a{}^b{}_{;b} = & -2F_{ae;f}F^{be;f}{}_b - 2F_{ae;fb}F^{be;f} + F_{ce;fa}F^{ce;f} \\
= & -2F_{ae;f}F^{be}{}_{;b}{}^f - 2F_{ae;f}\left(R^f{}_b{}^e{}_iF^{bi} - R^f{}_iF^{ie}\right) - 2F_{ae;bf}F^{be;f} - 2F^{be;f}\left(R_{fba}{}^iF_{ie} + R_{fbe}{}^iF_{ai}\right) \\
& + F_{ce;fa}F^{ce;f} \\
= & -2F_{ae;f}F^{be}{}_{;b}{}^f - 2F_{ae;f}\left(R^f{}_b{}^e{}_iF^{bi} - R^f{}_iF^{ie}\right) - 3F_{[ae;b]f}F^{be;f} + F_{eb;af}F^{be;f} \\
& - 2F^{be;f}\left(R_{fba}{}^iF_{ie} + R_{fbe}{}^iF_{ai}\right) + F_{ce;fa}F^{ce;f} \\
= & -2F_{ae;f}F^{be}{}_{;b}{}^f - 3F_{[ae;b]f}F^{be;f} + F_{eb;fa}F^{be;f} + 2F^{be;f}R_{afe}{}^iF_{ib} \\
& - 2F_{ae;f}\left(R^f{}_b{}^e{}_iF^{bi} - R^f{}_iF^{ie}\right) - 2F^{be;f}\left(R_{fba}{}^iF_{ie} + R_{fbe}{}^iF_{ai}\right) + F_{ce;fa}F^{ce;f}
\end{aligned} \tag{57}$$

Using the source-free Maxwell's equations (52) and rearranging gives

$$2\mathcal{C}_a{}^b{}_{;b} = \frac{1}{2}F_{ef;a}R^{ef}{}_{ib}F^{bi} + 2F_{ae;f}R^f{}_iF^{ie} + 2F^{be;f}R_{baf}{}^iF_{ie} - F^{be;f}R^i{}_{fbe}F_{ai} \tag{58}$$

and decomposing the Riemann tensor in four dimensions gives

$$\begin{aligned}
2\mathcal{C}_a{}^b{}_{;b} = & \frac{1}{2}F_{ef;a}C^{ef}{}_{ib}F^{bi} + 2F^{be;f}C_{baf}{}^iF_{ie} - F^{be;f}C^i{}_{fbe}F_{ai} \\
& + 2F_{af;e}S^f{}_iF^{ie} + 2F^{ie;f}S_{if}F^{ae} - F^{ie;f}S_{af}F^{ie} \ .
\end{aligned} \tag{59}$$

We have already noted that the 4-dimensional fddi in Lemma 3, $C_{[ab}{}^{[cd}\delta^{e]}_{f]} \equiv 0$, when contracted with $L_{de}{}^f$ gives the ddi (4); a further contraction with F^{cb} gives

$$\begin{aligned}
0 \equiv & 2F^{cb}L_{[b}{}^{de}C_{c]eda} - \frac{1}{2}F^{cb}L^{de}{}_aC_{decb} + 2F_a{}^bL^{def}C_{bdef} \\
& = 2F^{ie}L_e{}^{bf}C_{ifba} - \frac{1}{2}F^{ib}L^{fe}{}_aC_{feib} - F_a{}^iL^{bef}C_{ifbe} \ ,
\end{aligned} \tag{60}$$

and the substitution $L_{abc} = F_{ab;c}$ into (60) means that the first three terms on the right hand side of (59) disappear. Next we use Einstein's equations and equate the trace-free Ricci tensor S_{ab} to the usual expression for the electromagnetic energy tensor

$$S_a{}^b = T_a{}^b = F_{ae}F^{be} - \delta^b_aF_{cd}F^{cd}/4. \tag{61}$$

Then the last three terms on the right hand side of (59) can be rearranged to give

$$\left(F_{[b}{}^{b}F_{c}{}^{c}F_{d}{}^{d}F_{e}{}^{e}\delta_{a]}^{f}\right)_{;f} \equiv 0 \tag{62}$$

since the expression inside the brackets is identically zero in four dimensions by virtue of Lemma 1, which in this context is equivalent to the algebraic Rainich identity $T^{a}{}_{c}T^{c}{}_{b} = \delta_{a}^{b} \, T_{ij}T^{ij}/4$ where the energy-momentum tensor $T^{a}{}_{b} = F^{a}{}_{c}F^{c}{}_{b} - \delta_{a}^{b} \, F^{ji}F_{ij}/4$, [15]. $\diamondsuit$

We note that, in the proof, 4-dimensional fddis have been used explicitly on two occasions ((60) and (62)), as well as a decomposition of the Riemann tensor in four dimension. This is why the spinor proof appears much simpler, since the corresponding calculations in spinors just never occur. The use of the Einstein equations for the electromagnetic energy momentum tensor (61) is an important component of the proof; the last three terms on the right hand side of (59) cannot be removed by other means, such as a 4-dimensional ddi which was the means used to remove the first three terms.

From the point of view of a direct generalisation of this method to higher dimensions, it is clear that the higher dimensional analogues, (e.g., the 5-dimensional fddi in Lemma 3) would not be sufficient to reduce the first three terms of (59) to zero; nor would the 5-dimensional Cayley-Hamilton theorem be sufficient to reduce the last three terms of (59) to zero. So it would appear that there is no *simple and direct* generalisation of Theorem 6 in higher dimensions; but of course this does not rule out more involved generalisations. The fact that the algebraic Rainich identity was used in this 4-dimensional proof would suggest that higher dimensional analogues would require higher dimensional algebraic Rainich identities; in five dimensions this has been shown to be a cubic identity in the energy-momentum tensor $T^{a}{}_{b}$.

It may be of interest to note that the result is actually true more generally for the trace of the superenergy tensor $\mathcal{T}[\nabla F]_{abcd}$.

8. New symmetric Bel-Robinson tensor generalisations in higher dimensions.

We have noted in Section 6 that although $\mathcal{T}[\hat{C}]_{abcd}$ is symmetric in four and five dimensions this result does not generalise to higher dimensions. However, the 5-dimensional fddi which established this result has a counterpart in other dimensions; so we now investigate whether we can obtain analogous symmetry properties for some other superenergy tensors in higher dimensions.

Lovelock [14] has pointed out that the n-dimensional counterpart of Lemma 3 is,

Lemma 5. In $n = 2p$ dimensions, the trace-free double (p,p)-form $V_{i_1 i_2 \ldots i_p}{}^{j_1 j_2 \ldots j_p} = V_{[i_1 i_2 \ldots i_p]}{}^{[j_1 j_2 \ldots j_p]}$ satisfies

$$V_{[i_1 i_2 \ldots i_p}{}^{[j_1 j_2 \ldots j_p} \, \delta_{i_{p+1}]}^{j_{p+1}]} = 0 \ . \tag{63}$$

This specialises in *six dimensions (and lower)* for a trace-free double $(3,3)$-form to

$$V_{[abc}{}^{[efg} \, \delta_{d]}^{h]} = 0 \ . \tag{64}$$

The more general results in [15] include

Lemma 6. In $n \leq 2p + 1$ dimensions, the trace-free double (p,p)-form $V_{i_1 i_2 \ldots i_p}{}^{j_1 j_2 \ldots j_p} = V_{[i_1 i_2 \ldots i_p]}{}^{[j_1 j_2 \ldots j_p]}$ satisfies

$$V_{[i_1 i_2 \ldots i_p}{}^{[j_1 j_2 \ldots j_p} \, \delta_{i_{p+1}}^{j_{p+1}} \delta_{i_{p+2}]}^{j_{p+2}]} = 0 \ . \tag{65}$$

This specialises in *seven dimensions (and lower)* for a trace-free double $(3,3)$-form to

$$V_{[abc}{}^{[fgh} \, \delta_{de]}^{ij]} \equiv 0 \ . \tag{66}$$

This fddi (66) is the analogue of the fddi (45) used to establish symmetry of the Bel-Robinson tensor in five and four dimensions. So we expect (66) to establish symmetry for some generalisation of the Bel-Robinson tensor such as a *trace-free* double $(3,3)$-form in *seven (and lower) dimensions*.

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Senovilla [3] has given a basic superenergy tensor for the double $(3,3)$-form $K_{abc}{}^{def} = K_{[abc]}{}^{[def]}$ in n-dimensions as

$$
\begin{aligned}
\mathcal{T}[K]_{abcd} = \Big(K_{apqcrs} K_b{}^{pq}{}_d{}^{rs} + K_{apqdrs} K_b{}^{pq}{}_c{}^{rs} &- \frac{1}{3} g_{ab} K_{pqrcst} K^{pqr}{}_d{}^{st} - \frac{1}{3} g_{cd} K_{apqrst} K_b{}^{pqrst} \\
&+ \frac{1}{18} g_{ab} g_{cd} K_{pqrstu} K^{pqrstu} \Big)/4 \ ,
\end{aligned}
\tag{67}
$$

To keep things simple, and maintain the analogy with the Weyl candidate $\hat{C}_{ab}{}^{cd}$, we assume also that $K_{abc}{}^{def}$ is *trace-free and (block) symmetric*, i.e.,

$$
K_{abc}{}^{dea} = 0, \qquad K_{abc}{}^{def} = K^{def}{}_{abc} \ .
\tag{68}
$$

So $\mathcal{T}[K]_{abcd}$ clearly has the symmetry properties in n-dimensions,

$$
\mathcal{T}[K]_{abcd} = \mathcal{T}[K]_{(ab)cd} = \mathcal{T}[K]_{ab(cd)} = \mathcal{T}[K]_{cdab}
\tag{69}
$$

To determine if $\mathcal{T}[K]_{abcd}$ is symmetric in *all* indices we examine

$$
\begin{aligned}
\mathcal{T}[K]^a{}_{[bc]}{}^d = \Big(K^a{}_{pq}{}^d{}_{rs} K_{[b}{}^{pq}{}_{c]}{}^{rs} + K^{apq}{}_{[c|rs|} K_{b]pq}{}^{drs} &- \frac{1}{3} \delta^a_{[b} K^{|pqr|}{}_{c]st} K_{pqr}{}^{dst} \\
&- \frac{1}{3} \delta^d_{[c} K^{apq}{}_{|rst|} K_{b]pq}{}^{rst} + \frac{1}{18} \delta^a_{[b} \delta^d_{c]} K_{pqr}{}^{stu} K^{pqr}{}_{stu} \Big)/4 \ ,
\end{aligned}
\tag{70}
$$

The structure of (70) (including two deltas) suggests that we exploit the *seven dimensional identity* (66) and investigate

$$
0 \equiv K_{[efg}{}^{[hij} \delta^{ad]}_{bc]} K^{efg}{}_{hij} \ .
\tag{71}
$$

When we write out (71) as

$$
\begin{aligned}
0 \equiv \Big(\frac{1}{3} K_{bcp}{}^{qrs} K^{adp}{}_{qrs} - \frac{1}{2} K_{pqc}{}^{fgd} K^{pqa}{}_{fgb} + \frac{1}{2} K_{pqc}{}^{fga} K^{pqd}{}_{fgb} &- \frac{1}{6} \Big(K_{pqi}{}^{fga} K^{pqi}{}_{fgb} \delta^d_c \\
- K_{pqi}{}^{fga} K^{pqi}{}_{fgc} \delta^d_b + K_{pqi}{}^{fgd} K^{pqi}{}_{fgc} \delta^a_b - K_{pqi}{}^{fgd} K^{pqi}{}_{fgc} \delta^a_c \Big) &+ \frac{1}{18} K^{pqr}{}_{fgh} K_{pqr}{}^{fgh} \delta^{ad}_{bc} \Big)/4
\end{aligned}
\tag{72}
$$

we easily see that the right hand side of (72) does not coincide with (70), because of an apparent discrepency in the respective first terms. (Using the symmetry properties (68) enables us to match up all the other terms.)

However, if we consider K_{abcdef} to satisfy a *first Bianchi-type identity*

$$
K_{ab[cdef]} = 0
\tag{73}
$$

— as well as being trace-free and (block) symmetric — then we find that the first term on the right hand side of (70) becomes

$$
K^a{}_{pq}{}^d{}_{rs} K_{[b}{}^{pq}{}_{c]}{}^{rs} = -K^a{}_{pqrs}{}^d K_{bcpqrs} = K^{ad}{}_{pqrs} K_{bc}{}^{pqrs}/3
\tag{74}
$$

and now it is easy to see that the right hand side of (72) coincides term by term with (70), and so we have proved

Theorem 7. In seven (and less) dimensions a trace-free symmetric double $(3,3)$-form $K_{abc}{}^{def}$ which also satisfies $K_{ab[cdef]} = 0$ has a superenergy tensor $\mathcal{T}[K]_{abcd}$ given by (67) which is symmetric in all indices.

$$\diamondsuit$$

Again, in analogy with the Bel-Robinson tensor in four dimensions we can see from a direct calculation on the index pair (ab) in (67), combined with Theorem 7, that

Corollary 7.1. In six dimensions, the trace-free symmetric double $(3,3)$-form $K_{abc}{}^{def}$ which also satisfies $K_{ab[cdef]} = 0$, has a superenergy tensor $\mathcal{T}[K]_{abcd}$, given by (67) which is symmetric in all indices and trace-free on all pairs of indices.

$$\diamondsuit$$

Let us consider K_{abcdef} to be also *divergence-free*, i.e.,

$$K^a{}_{bcdef;a} = 0 \ . \tag{75}$$

It is then straightforward to repeat the type of calculation which has been done for the divergence-free Weyl tensor in Ricci-flat spaces in n dimensions, and show that

Corollary 7.2. If in addition $K_{abc}{}^{def}$ also satisfies $K^a{}_{bcdef;a} = 0$ then its superenergy tensor $\mathcal{T}[K]_{abcd}$ is also divergence free in n dimensions., i.e.,

$$\mathcal{T}[K]^a{}_{bcd;a} = 0 \ . \tag{76}$$

$$\diamondsuit$$

We have given just this one application as a simple example to illustrate the power of the fddis in generalising a result to a higher dimension. However, from this pattern, we would expect to generalise Theorem 7 to classes of (p,p)-forms in spaces of dimension $n = 2p+1$. Similarly it appears likely that the result in Theorem 5 for Lanczos candidates in four dimensions, could be generalised to classes of (p,q)-forms in $n = p + q + 1$ dimensions.

9. Discussion.

We have used ddis in many situations in this paper, both to rederive existing results, in an efficient tensorial manner, and to obtain new results. The four quadratic identities, which we consider in Sections 2 and 3, are all special cases of one spinor result which can be established very easily in spinors. We were able to derive the corresponding tensor identities in a manner independent of signature. However, the tensor versions are of varying degrees of difficulty: the familiar Bel-Robinson identity in Theorem 2a requires considerable preliminary work to establish lemmas for the Weyl tensor, and Theorem 2d requires Theorem 2a together with a number of other lemmas, one involving a 'mixed' identity. The tensor derivation of the new conservation law for electromagnetism [5] given in Section 7 also requires 'mixed' ddis. These tensor derivations are very complicated, and one wonders how long it would have taken to even conjecture the results in Theorems 2d and 6 without the parallel spinor (or null vector) results; but it seems that there is no easier signature-free way in tensors, and we should train ourselves to recognise such structures in tensors. Of course, none of the fddis which we are using in this paper are 'new', but while ddis which involve only one tensor (such as quadratic identities for the Weyl tensor, or the Cayley-Hamilton theorem for the Ricci tensor), are familiar, on the otherhand, 'mixed' ddis such as (4), (35), (60), (62) involving more than one tensor and/or derivatives are much less familiar; one of the purposes of this paper is to draw attention to such possibilities.

The amount of work required in these tensor calculations in four dimensions serves as a warning of the even more complicated calculations which will be required to establish analogous results in higher dimensions; the existence of fddis as 'signposts' will be invaluable.

It is clear from the above examples that the exploitation of dimensionally dependent identities is a useful method, not only for confirming suspected tensor identities, but also for establishing new and perhaps unexpected results. For instance, when we have a particular tensor expression, a study of its structure can suggest an overlap between some of its terms and the terms in a dimensionally dependent identity, and so we have an opportunity of exploiting the latter, and discovering hitherto unexpected tensor relationships. (This is actually how the unexpected symmetry property of the Lanczos superenergy tensor was first recognised in [15].) Furthermore, once a new significant tensor relationship is established in four dimensions and the kernel 4-dimensional fddi identified, then the analogous fddis in five dimensions and higher can be investigated with the hope of establishing new tensor relationships in these higher dimensions.

Theorem 7 is an example of how to exploit this approach. We can continue to look for higher dimensional analogues of significant 4-dimensional results: the identity (63) for the trace-free double (p,p)-form $V_{i_1 i_2 \ldots i_p}{}^{j_1 j_2 \ldots j_p} = V_{[i_1 i_2 \ldots i_p]}{}^{[j_1 j_2 \ldots j_p]}$ in dimensions $n = 2p$ leads to the quadratic identity [14]

$$V_{x i_2 \ldots i_p}{}^{j_1 j_2 \ldots j_p} V_{j_1 j_2 \ldots j_p}{}^{y i_2 \ldots i_p} = \delta_x^y \ V_{i_1 i_2 \ldots i_p}{}^{j_1 j_2 \ldots j_p} V_{j_1 j_2 \ldots j_p}{}^{i_1 i_2 \ldots i_p}/2p.$$

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The quadratic identity (1) for the Bel-Robinson tensor (i.e., the superenergy tensor for the Weyl tensor where $p = 2$) in four dimensions motivates the question as to whether there exists an analogous quadratic identity for the superenergy tensor of (the block symmetric part of) $V_{i_1 i_2 \ldots i_p}{}^{j_1 j_2 \ldots j_p}$ in dimension $n = 2p$. The stronger versions, and the higher dimensional generalisations will be presented elsewhere. One criticism of this new generalised Bel-Robinson tensor $\mathcal{T}[K]_{abcd}$ would be the apparent lack of explicit connection of $K_{abc}{}^{def}$ with physical fields; we shall demonstrate in a future paper that there are indeed important links with the gravitational field as described by the Weyl tensor C_{abcd}, and that we can construct examples of the tensor $K_{abc}{}^{def}$ which inherit some of the properties of C_{abcd}.

The result in Theorem 7 has illustrated one possible way to exploit analogous fddis in higher dimensions: to generalise to forms of higher rank. Another approach has been taken in the generalised Rainich problem [16], where higher dimensional identities analogous to 4-dimensional quadratic identities were obtained as cubic and higher order identities — involving very long manipulations. It seems clear from the tensor derivation of the new conservation law in electromagnetic theory, that *direct* generalisation to higher dimensions are not possible. However, we would expect some sort of generalisation exploiting the higher dimensional counterparts of the kernel fddis used in four dimensions. With the generalised Rainich results [16] as 'signposts' we would speculate there may be a generalisation of the new conservation law in five dimensions for superenergy tensors involving *cubic* terms of T_{ab}.

While obviously interesting and useful in themselves, the study of these identities is not an end in itself. What is of interest is to find identities which are sufficient as well as necessary conditions for a factorisation result, and to be able to study a 'generalised Rainich-Maisner-Wheeler problem'. Clearly then we will need the most general versions of such identities; for instance Penrose [2] has given the most general spinor version of the spinor identity (12) for the Bel-Robinson tensor — a quadratic identity with *all* indices free — and shown that it is also sufficient to achieve the factorisation (11a). The ddis which we have been studying in this paper will give us the basic structures to continue these investigations.

Finally, we would also emphasise how simple it is to disprove conjectured identities by counterexample; *GRTensorII*, [25] is an invaluable tool for this, and it can also be efficient in enabling us to distinguish between trivial and non-trivial identities. We have also benefited from the use of *Tensign* [28], making it possible to guarantee the accuracy in the extensive index manipilation required in some of the results.

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